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FRACTAL ROUGHNESS IN CONTACT PROBLEMS†

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The roughness of real polished bodies is shown to be fractal in character. A relation is found between the fractal dimension of a surface and its statistical properties. Models are constructed of the contact of fractal-rough punches and the smooth surface of a deformable half-space by a modelling Winkler medium and a rigidly plastic medium. At the macrolevel, the impressed punches are regarded as either flat or convex. At the initial stage of indentation, when the proximity of the punch and the medium is much less than the size of irregularities, asymptotic power laws have been obtained which associate the force operating on the punch and the depth of indentation for different (both plastic and elastic) models of the deformed base. The relation between the power index and the fractal dimension of the surface and the print is determined.

IT IS well known [1–3] that the parameters of the actual contact zone of real bodies depend closely on the waviness and roughness of the contact surfaces. The analysis of the effect of roughness on the contact interaction parameters has attracted wide attention [1–5]. The associated problems primarily arise from the fact that an exact analytic solution of the contact problems of the theory of elasticity and plasticity is only known for a few cases of bodies with a regular geometric shape. For this reason, allowance for surface roughness must be made using more or less simple models, such as regular sinusoidal surface roughness (or waviness) [3], in which the contact zones at low pressures are determined from the Hertz theory. At high pressures, the interfaces are assumed to be small and can be found in the model of a crack under pressure. It was shown in [6] that the nature of the approach of two elastic half-spaces with an interface of small area between their boundaries depends essentially on the original geometry of the boundary, and so the application of the model of a crack for the contact problem needs additional explanations.

It stands to reason that the microgeometry of the surface of bodies will have a large effect on their contact properties, especially at the initial stage of compression.

A remarkable feature of the surface structure of many actual bodies has been discovered recently: it turns out that the surface irregularity of bodies can very often be characterized as self-similar, (that is, it exhibits scaling behaviour (in a statistical sense) during a scalar transformation (stretch transformation of the coordinates) [7–12]. This means, in particular, that, at least on the mesoscopic scale (that is, on a scale between the microscopic and macroscopic levels), a surface has irregularities of all scales. It seems very natural to use the apparatus of the theory of fractal sets (fractal surfaces) [7, 13, 14] to construct a geometrical model of a real surface with the above properties because, as we know, fractal surfaces possess self-similarity or self-affinity and are characterized by the presence of a cascade of irregularities of all scales.

Roughness was apparently first modelled by imposing self-similar distortions of smaller scale on an originally smooth convex surface in [15] (see also [2]). Note that the surface of the Archard model is not a fractal surface.

As we have said, the methods of fractal theory are closely connected with similarity transformations. It is obvious, though, that the use of similarity methods in contact problems might be unconnected with fractals [16]. The fact that the roughness of real bodies is fractal in character [12]

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explains the attempts to allow for fractal behaviour in tribiological problems [9, 17, 18]. It is worth noting that the model constructed in [19]† allows for self-similarity of discrete contact between real bodies by describing the self-similarity effect using the technique employed in damage accumulation during cyclical loading of solids [20]. It was shown in [21] that some of the results obtained in [20] can be rephrased in the language of fractal theory.

1. THE FRACTAL DIMENSION OF A SURFACE AND A STATISTICAL ANALYSIS OF ITS GEOMETRY

We recall that a fractal is usually defined as a set for which the Hausdorff–Besicovitch dimension exceeds the topological dimension [7].

The dimension D_H of a Hausdorff–Besicovitch set S is defined as the critical dimension at which the d-dimensional measure $M_d(S)$ of a set S (d-dimension of covering S) changes from zero to infinity [7, 14]

$$M_d(S) \sim N(\delta)\delta^d|_{\delta\to 0} \to \begin{cases} 0, & d > D_H \\ \infty, & d < D_H \end{cases} \tag{1.1}$$

Here $N(\delta)$ is the number of sets of size δ that cover S.

For example, when a plane square is covered by segments ($d = 1$), $M_1 \sim \delta^{-1} \to \infty$ as $\delta \to 0$, but when it is covered by cubes ($d = 3$) $M_3 \sim \delta \to 0$, and only for $d = 2 = D_H$ do we obtain a finite measure M_2.

It follows from (1.1) that the dimension D of a fractal (identified with D_H in this case) is defined by the formula [7]

$$D = \lim_{\delta\to 0} \inf \frac{\ln N(\delta)}{\ln(1/\delta)} \tag{1.2}$$

Obviously, D is not necessarily an integer, and so sets with fractional dimensions can also be fractals. For fractal surfaces (topological dimension $d = 2$) $2 < D \leqslant 3$. It is usual to say that D characterizes the degree of waviness and ruggedness of a surface. When $D = 2$, it is a normal "smooth" surface, and when $D = 3$, the surface is rugged and crinkled so that, in fact, it "fills" a layer in three-dimensional space.

From the definition of a fractal, it follows that as the characteristic size of the covering δ (scale of "measurement" of the fractal) tends to zero, the characteristic linear measure of the fractal tends to infinity. Thus, for a fractal line ($D > 1$) the length $L(\delta)$ measured with accuracy δ diverges as

$$L(\delta) \sim N(\delta)\delta \sim \delta^{1-D}|_{\delta\to 0} \to \infty \tag{1.3}$$

Similarly, for the area of a fractal surface, we have

$$S(\delta) \sim N(\delta)\delta^2 \sim \delta^{2-D}|_{\delta\to 0} \to \infty \tag{1.4}$$

The properties (1.3) and (1.4) dictate the need for restrictions on the limits of applicability of the fractal model

$$\delta_* \ll \delta \ll \Delta_*$$

In fact, the waviness of a fractal surface is infinite, whereas that of a real surface, of course, is finite. It is therefore natural to introduce a lower limit δ_* of applicability of a fractal model. The value of δ_* is usually associated with the microstructure of the material; for metals it could be the size of a grain, subgrain or any other characteristic structural element. On the other hand, there might also be an upper limit of applicability of the fractal model Δ_*, connected with the geometrical dimensions of the body, the characteristic scale of non-uniformity of external fields, etc.

† See also BORODICH F. M., Experimental investigations of the compression of multilayer metal packets. Unpublished paper, Moscow, 1985. Deposited at VNIIIS Gosstroya SSSR. 12.09.85. No. 6223–85.

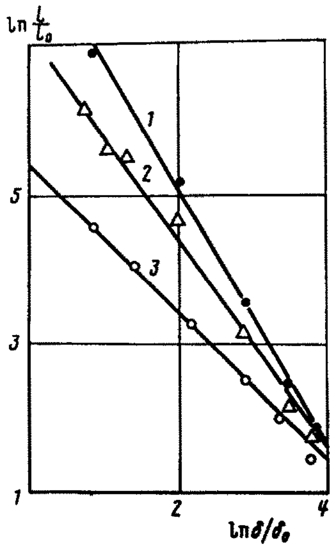

FIG. 1.

The fractal nature of the surface of real bodies has been confirmed by the results of numerous experimental investigations [7, 8, 11, 12, 22–24]. The surface of many porous and ceramic materials, of cracks in metals and rocks, the surface of metals after mechanical treatment and so on, have all been found to be fractals. A number of techniques, based on different physical principles, have been proposed for finding the fractal parameters of the surface of solids. The most conventional are methods based on an analysis of profiles (using (1.2)), and the "island" method of Mandelbrot [11], based on computer analysis of images obtained when a surface is intersected by a horizontal surface. In other methods, use has been made of acoustic measurements [22], adsorption experiments [8, 23], and the results of neutron and X-ray scattering by a surface [24].

We will give an example of finding the fractal dimension of a surface by an analysis of profiles. In this case the result [25] that when a stochastic fractal is intersected by a plane in a general position, the resulting set is a fractal of dimension $D-1$ is commonly used.

By considering the profile from this point of view and, using (1.2), finding its dimension D_F, the dimension of the surface can be regarded (with probability one) to have dimension $D = D_F + 1$.

The dependence of the length L of the profile of polished surfaces of metals after plane-polishing on the measurement scale δ plotted from profile data [1] is shown in Fig. 1.

The results show that the roughness of the polished surface of a metal is a fractal, and the dimension D_F of the fractal depends on the purity of the treatment. Curves 1–3 correspond to the following surfaces: ninth grade accuracy, $D_F \approx 1.8$; reduction of the same surface, tenth grade accuracy, $D_F \approx 1.5$; same, but now fourteenth grade, $D_F \approx 1.0$. The graphs were plotted for relative lengths L/L_0 and δ/δ_0, and the points correspond to experimental data. The lines are drawn by the method of least squares.

As already mentioned, fractal structures are locally invariant under a scalar transformation, and for a stochastic fractal, all directions are equally likely and the extension factor is the same for all coordinates. This is quite acceptable for describing structures that formed under isotropic conditions (the various fractal clusters give a clear example of this [14]). But if the fractal structures form under circumstances where there is appreciable anisotropy of the properties or processes, isotropic behaviour during a scalar transformation can no longer be expected. However, an attempt can still be made to keep the basic idea of the fractal approach, associated with scaling of the structure during scalar transformations, using a group of self-affine scalar transformations (quasi-homogeneous extension) $\mathbf{x} \to (\lambda_1 x, \lambda_2 y, \lambda_3 z)$ where, generally speaking, $\lambda_1 \neq \lambda_2 \neq \lambda_3$, instead of a simple dilatation group (homogeneous extension with respect to all the coordinates) $\mathbf{x} \to \lambda \mathbf{x}$. This yields the important concept of self-affine fractals [7, 26, 27], which are locally invariant (for stochastic fractals, in terms of their distribution) in respect of self-affine transformations. The properties of self-affine fractals are immeasurably richer than those of self-similar fractals, although the two have much in common. Without dwelling on this question (see [27], for example), we will

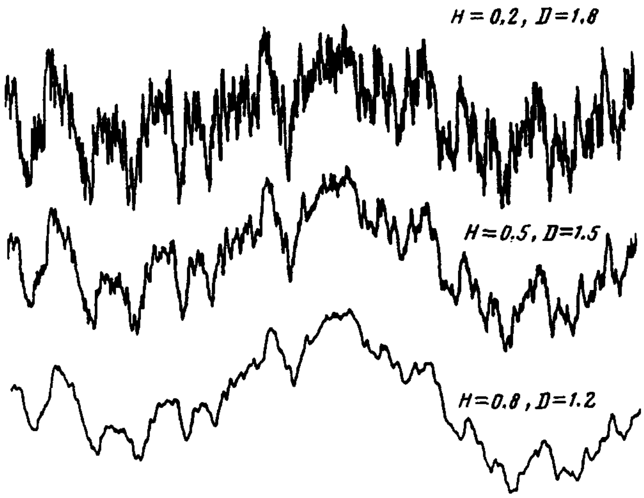

FIG. 2.

examine self-affine fractal structures associated with fractional Brownian motion, which are important for the discussion that follows.

In the one-dimensional case, a fractional Brownian process $B_H(t)$ is a single-valued random function of the variable t, such that the increment $B_H(t_2) - B_H(t_1)$ has a Gaussian distribution with variance

$$\langle | B_H(t_2) - B_H(t_1)|^2 \rangle \sim |t_2 - t_1|^{2H}, \quad 0 < H < 1 \tag{1.5}$$

Here and below angle brackets denote averaging over the ensemble.

When $H = ½$ we obtain the usual Brownian process with independent increments for which $\langle \Delta B^2 \rangle \sim \Delta t$. When $H \neq ½$, the properties of a fractional Brownian process are very different from usual, since there is positive correlation between the increments when $H > ½$ and negative correlation when $H < ½$. In the latter case, the correlation effect can extend to large intervals of time and can have a substantial influence on the shape of the process trajectories. Figure 2 shows the example of trajectories of a fractional Brownian process for different [27].

From (1.5) we find that when the scale of the variable is changed: $t \to \lambda t$ (i.e. $\Delta t = t_2 - t_1 \to \lambda \Delta t$), the increment changes according to the rule $\Delta B_H \to \lambda^H \Delta B_H$. Thus, when considering the trajectory $B_H(t)$ in the t, B plane, it is reasonable to talk of the self-affine behaviour of $B_H(t)$.

It can be shown that the trajectories of a fractional Brownian process are self-affine fractals with dimension $D = 2 - H$.

A fractional Brownian process in d-dimensional space can be defined as before. In this case, instead of (1.5), the variance must be written in the form

$$\langle | B_H(\mathbf{x}_2) - B_H(\mathbf{x}_1)|^2 \rangle \sim |\mathbf{x}_2 - \mathbf{x}_1|^{2H} \tag{1.6}$$

The fractal dimension of the trajectories of a process of this kind is defined as: $D = d + 1 - H$.

We shall be interested below in self-affine fractal surfaces $z = z_H(x, y)$ which are modelled by fractional Brownian motion. In this case $d = 2$, $\mathbf{x} = (x, y)$, $B_H = z_H$ and, correspondingly, $D = 3 - H$.

An appropriate way of characterizing the fractal properties of a surface is to use the spectral density $S_z(k)$, defined as follows:

$$\langle z_H(\mathbf{x})\, z_H(\mathbf{x} + \boldsymbol{\delta}) \rangle = \int S_z(\mathbf{k}) \cos(2\pi \mathbf{k} \cdot \boldsymbol{\delta})\, 2\pi \mathbf{k}\, d\mathbf{k} \tag{1.7}$$

Since all directions in the x, y plane are assumed to be equivalent, $S_z(k)$ will depend only on $k = |\mathbf{k}|$ and it can be shown [26] that

$$S_z(k) \sim k^{-(\beta + 1)}, \quad \beta = 1 + 2H \tag{1.8}$$

Thus, if we know the power scalar behaviour of the spectral density S_z, we can find the fractal dimension of a surface from the rule $D = (7-\beta)/2$.

Investigations of the rough surface of real bodies after friction or mechanical treatment show in many cases that the irregular structure exhibits scaling behaviour as the scale of observation is changed [9, 10, 12]. It should be remembered, however, that the processes which give rise to the surface irregularities are usually highly anisotropic—the direction "along" the surface is very different from the transverse direction. This means that the correlation lengths "along" and "across" the surface can differ greatly both in value and in the properties revealed during scaling, although self-similar behaviour of the surface (in the self-affine sense) is preserved.

We will model a self-affine rough surface with the help of fractional Brownian motion. If the correlations are mainly longitudinal, then $H<½$, if they are mainly transverse, then $H>½$. Relations (1.6)–(1.8) provide a way of finding the fractal parameters of a surface from a statistical analysis of its geometry, as follows. After analysing the profiles, we construct the correlation function (1.7) and use it to find the spectral density $S_z(k)$. Then, plotting the relation $\ln S_z(k) \sim \ln k$, we see that the experimental points lie on a straight line, the slope of which defines β and, therefore, H and D. But if $S_z(k)$ in logarithmic coordinates is not linear, the simple fractal model does not describe the roughness of that surface, and an attempt can be made to construct a model using a multifractal measure [7, 14]. This case will not be considered here.

2. SIMILARITY IN THE MODELLING OF THE CONTACT OF ROUGH BODIES AND THE FRACTAL CONTACT MODEL

In this paper we are mainly interested in the problem of the indentation of a polished fractally rough body in a half-plane or half-space. Of course, it is quite hopeless to expect to obtain the exact solution of a contact problem for a rough body whose surface is being modelled, for example, using a fractional Brownian process. However, we need merely use observations based on analytic and numerical investigation of the many processes that are fractal in nature. In most cases it turns out that the specific character of a fractal model has very little effect on the asymptotic behaviour of the processes, and it is its dimension (fractal, spectral, etc.) which is of most importance. Although this statement has not been rigorously proved, it seems very reasonable and can be used to obtain important results on the asymptotic behaviour of the solution of many fractal problems. Below we select a surface constructed on the basis of a regular fractal and analyse the contact problem for that surface, assuming that the results hold for all problems with surfaces of a punch and indentation of the same fractal dimension.

Possibly the simplest model of a polished fractal surface is based on a Cantor set [14, 28]. By joining segments obtained at successive stages of the construction of a Cantor set to one another, we obtain the object shown in Fig. 3. At each successive stage of the construction of the set, the middle of each initial segment is discarded, so that the total length of the remaining segments is $1/a$ of the

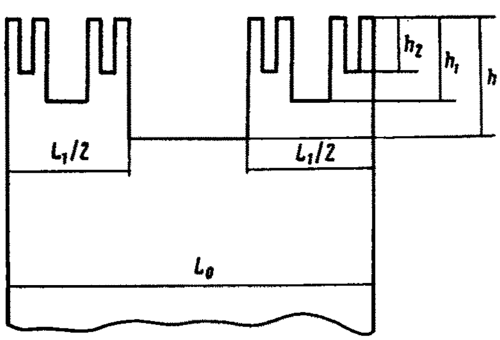

FIG. 3.

original ($a>1$), the depth of the indentations (measured from the top of the structure) is b times ($b \geq 1$) smaller in the $n+1$th step than the nth step. The surface is assumed to be smooth in a direction perpendicular to the plane of the page. This is not a very important restriction. It will be shown below that it is also possible to construct a fractal Cantor surface perpendicular to the plane of the page.

The impression might be gained that this model of a surface is completely unrealistic. Of course, it cannot be expected to be able to describe the surface roughness of a real body by a Cantor set. This model has been used in order to conduct an exact analytic investigation of the solutions of contact problems, but nevertheless it might turn out not to be too far from reality, since results on the actual contact area of polished (ground) metal surfaces show that they contain sets of parallel jagged-edged scratches of different depths [1, 3, 29].

For the fractal model of a surface, the scratches are assumed to be self-affine. We will investigate the model from the point of view of fractal geometry.

We will first find the dimension D_C of a plane Cantor set (Fig. 3). The section of the nth generation of this set will contain $N=2^n$ segments, each of length $l_n=(2a)^{-n}L_0$, where L_0 is the length of the null generation. If the section of the set is covered by N segments of length $\delta=l_n$, all points of the section will be covered.

From (1.2) we obtain the dimension of the section of a Cantor set

$$D_C = \ln N(\delta)/\ln(1/\delta) = \ln 2/\ln 2a$$

If a Cantor set is embedded in a half-space, we obtain a "print", which is also a fractal. Since the print is obtained as the result of the direct product of the Cantor set C and the straight line L, we have

$$N_{C \otimes L} = \delta^{-D_C}\delta^{-1}$$

From this we obtain the dimension of the print $D_0 = 1 + D_C$.

Let us see under what conditions the surface constructed is itself a fractal (cf. Fig. 3). After n iterations, the length of the contour is equal to

$$L_{(n)} = L_0 + 2h(q^{n+1} - 1)/(q-1), \quad q = 2/b$$

where L_0 is the width of the body and h is the depth of the largest indentation on its surface.

It is obvious that if $q<1$, the length of the contour is finite ($L_{(n)}$ tends to a finite limit as n increases). Hence, only for $b \leq 2$ ($q \geq 1$) will the contour of the surface of a body be a fractal.

We will find the dimension of this fractal D_F.

For $b<2$ and fairly large n, we have $L_{(n)} \sim 2hq^{n+1}/(q-1)$. Hence

$$L_{(n+1)}(l_{n+1}) \approx qL_{(n)}(l_n), \quad l_{n+1} = l_n/(2a) \tag{2.1}$$

where l_n is the scale of measurement.

From (1.2) and (2.1) it follows that

$$D_F = 1 + \frac{\ln 2}{\ln 2a} - \frac{\ln b}{\ln 2a} = 1 + D_C - \frac{\ln b}{\ln 2a} \leq 2 \tag{2.2}$$

The dimension D_S of the whole fractal surface, as already mentioned in Sec. 1, is equal to $D_S = 1 + D_F$. This result is quite consistent with the definition of the fractal dimension of a self-affine fractal given in Sec. 1.

In order to show this, we need to calculate the index H for the model. Of course, this value is purely nominal in the case of a regular fractal but it is not difficult to obtain under certain assumptions.

We will consider the profile of the surface (Fig. 3) as the graph of a certain step function $y=y(x)$. It can be seen that with scaling $\Delta x \to (2a)^{-1}\Delta x$ corresponding to an iterational step in constructing the surface, fluctuations Δy of y behave as $\Delta y \to (ab)^{-1}\Delta y$. In fact, at the nth iteration step $\Delta y_n \sim y_n p(y_n)$, where $P(y_n)$ is the probability of obtaining the value y_n, $y_n = h/b^n$, $p(y_n) = (1-a^{-1})a^{-n}$.

Putting $\Delta y \sim \Delta x^H$, we find $(2a)^H = ab$. This gives

$$H = \ln(ab)/\ln(2a) \tag{2.3}$$

By the definitions of Sec. 1, the fractal dimension $D = 2 - H$, and from (2.3) this is the same as (2.2).

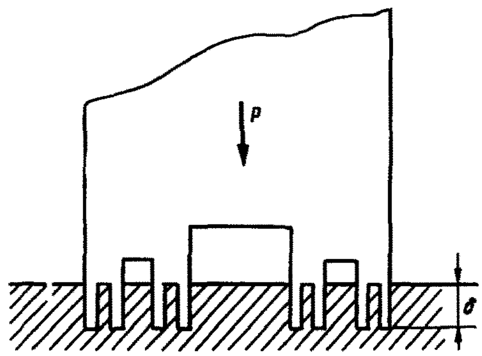

FIG. 4.

3. INDENTATION OF A PLANE FRACTAL PUNCH

We will consider the indentation of a macroscopic plane punch of width L_0 with the Cantor surface constructed above on a smooth half-space, as shown in Fig. 4. We will first examine the plastic problem, using the model of a rigid plastic body with plasticity limit during shear τ_Y.

To solve the problem of the indentation of a fractal punch, wide use will be made of Hill's solution of a punch with a plane base [30]. The limit load per unit length of the punch is equal to $P_* = \tau_Y L_0(2+\pi)$. Each protrusion of generation n of the fractal punch is effectively a plane Hill punch. To eliminate the effect of protrusions of one generation on another, it must be assumed that $a > 3/2$.

Let P_{n+1} be the limit for protrusions of the $(n+1)$th generation. We shall assume that, when that limit load is reached, the punch presses into the half-plane to a depth $\Delta\alpha_{n+1}$, equal to the difference between the heights of protrusions of the nth and $(n+1)$th generations.

These assumptions are sufficient to determine the dependence of the limit load P on the indentation depth α.

We will use the fact that the punch is impressed by an amount $\Delta\alpha_{n+1}$ when the limit load increases from P_{n+1} to P_n. Then

$$\begin{aligned} &\Delta P_{n+1} = P_n - P_{n+1} = \tau_Y(2+\pi)(L_n - L_{n+1}), \quad L_n = a^{-n}L_0 \\ &\Delta\alpha_{n+1} = \alpha_n - \alpha_{n+1} = bhb^{-(n+1)}(b-1) \\ &\frac{\Delta P_{n+1}}{\Delta\alpha_{n+1}} = \tau_Y(2+\pi)\frac{a-1}{b-1}\frac{L_0}{bh}\left(\frac{b}{a}\right)^{n+1} \end{aligned} \tag{3.1}$$

Since $\alpha_n = bhb^{-n}$, from (3.1) in the limit as $n \to \infty$

$$\frac{dP}{d\alpha} = \tau_Y(2+\pi)\frac{a-1}{b-1}\frac{L_0}{bh}\left(\frac{\alpha}{bh}\right)^{\chi-1}, \quad \chi = \frac{\ln a}{\ln b}$$

Thus, the dependence of the force P on the depth of indentation α of the punch in the medium has the form

$$P = P_* \frac{a-1}{b-1}\chi^{-1}\left(\frac{\alpha}{bh}\right)^{\gamma_1}, \quad \gamma_1 = \chi = \frac{1-D_C}{1+D_C-D_F} \tag{3.2}$$

For the plastic indentation of rough bodies the exponent $\gamma < 1$ [1]. In the case here, this condition will hold if

$$3/2 \leqslant a < b \leqslant 2$$

We now consider the problem of the indentation of a fractal punch in the elastic formulation.

It was shown in [31], using asymptotic methods, that when an elastic half-space is covered by a thin elastic layer (operating like Kirchhoff–Love plate) of equal or lower rigidity, in a number of cases the thin covering can operate like a layer of Winkler springs, with the elastic base operating like a continuous set of vertical elastic rods (or springs) [3, 32].

We will therefore examine the indentation of a fractal punch into a Winkler base of depth η. Using the same reasoning as before (here E is the modulus of elasticity of the base), we have

$$\Delta P_{n+1} = P_n - P_{n+1} = L_n \Delta\alpha_{n+1},$$
$$\Delta P_{n+1}/\Delta\alpha_{n+1} = E\eta^{-1} L_0 a^{-n}, \quad \eta = bh \tag{3.3}$$

In the limit as $n \to \infty$ after integration we obtain

$$P = \frac{EhL_0}{\eta(1+\chi)} \left(\frac{\alpha}{h}\right)^{\gamma_2}, \quad \gamma_2 = \frac{2 - D_F}{1 + D_C - D_F} > 1 \tag{3.4}$$

Note that the non-linear relation (3.4) merely assumes fractal roughness of the punch, and not non-linear compliance of the base.

Thus, when rough punches with a fractal surface are used, asymptotic power laws (3.2) and (3.4) are obtained for the dependence of the load on the depth of the indentation. The exponents γ_1 and γ_2 here depend explicitly on the fractal dimension of the "print" and the contour of the punch. With the initial ("elastic") depth of indentation, $1 < \gamma_2 < 2$, but in the plastic case $0 < \gamma_1 < 1$. The change in the bend of the curve is consistent with the available experimental data (see, for example, the results on lead [1] and on steel (the latter are given in the paper referred to in the previous footnote).

4. INDENTATION OF A CONVEX FRACTAL PUNCH

We will consider the case of plane deformation, for a convex, rather than a plane punch with a fractal surface, with $z = f(x)$, where f is the profile function. Let α be the depth of indentation of the point of the punch. Without loss of generality, it can be assumed that the profile of the punch is symmetrical about the z axis.

In the two models of an elastic and plastic base considered above, the contact pressure under the punch at any point depends only on its displacement. The contact force (by virtue of the symmetry of the shape) is given by the formula

$$P = \int_0^{l_*} p(x)\,dx, \quad f(l_*) \equiv \alpha$$
$$p(x) = c_i[\alpha - f(x)]^{\gamma_i}, \quad i = 1, 2 \tag{4.1}$$

where, from (3.2) and (3.4), we have

$$c_1 = \tau_Y(2+\pi)\frac{a-1}{b-1}\chi^{-1}(bh)^{\gamma_1}, \quad c_2 = Eh^{1-\gamma_2}\eta^{-1}(1+\chi)^{-1}$$

for a plastic base ($i = 1$) and an elastic base ($i = 2$).

Let $f = Ax^m$, $A > 0$, $m > 1$. Then from (4.1) we have

$$P = c_i \int_0^{l_*} [\alpha - Ax^m]^{\gamma_i} dx = c_i \alpha^{\gamma_i + 1/m} \int_0^{A^{-1/m}} [1 - A\xi^m]^{\gamma_i} d\xi = O(\alpha^{\gamma_i + 1/m})$$

For a rough convex punch (with macroscopic profile $z = A\rho^m$, $\rho = \sqrt{x^2 + y^2}$), impressed into a Winkler medium, it can be shown that

$$P \sim \alpha^{\gamma_3 + 2/m}$$

For a smooth punch, we have $D_F = 2$, $D_0 = 2$, $\gamma_3 = 1$. Then $P \sim \alpha^{1+2/m}$, in complete agreement with known results [3].

Thus, it has been shown in a specific case that fractal roughness of the surface of contacting bodies gives the well-known power laws of the form $P \sim \alpha^{\gamma}$.

The exponent γ turns out to be closely connected with the fractal parameters of the contacting pair: the fractal dimensions of the rough surface and the indentation. Presumably, from the universality of fractal objects [7, 14], the exponent γ in the general case too is governed only by the macroscopic characteristics of the fractal surface (such as its dimension), and the structural detail of the fractal is unimportant. It is in this sense that the exponent γ can be said to be universal. On the other hand, the multiplier A_0 in the power dependence $P \sim A_0 \alpha^{\gamma}$ will include information on the structural detail of the surface and will therefore depend on the specific shape of the contacting surfaces.

5. THE SPATIAL PROBLEM

We can extend these results to the three-dimensional case. We will start by constructing a regular self-affine rough surface. By analogy with the previous sections, the model of such a surface is constructed using a Cantor set [28, 33]. For simplicity, consider a plane punch of square cross-section of size $L \times L$. At the first stage in the construction of Cantor roughness, a cross (of thickness h) is cut out of the surface of the punch, dividing the surface into four congruent squares $(L/a) \times (L/a)$. Another cross (of thickness h/b) is then cut out of each of the remaining squares, leaving four squares $(L/a^2 \times L/a^2)$, and the process is repeated *ad infinitum*.

As a result, we obtain a self-affine fractal surface of dimension

$$D = 3 - H, \quad H = \ln(ba^2)/\ln(2a)$$

The indentation of the punch into the plane is two-dimensional Cantor "dust", a regular fractal of dimension $D_0 = \ln 4/\ln(2a)$.

As in Sec. 3, by considering the indentation of a rigid fractal punch with a self-affine rough surface into an elastic (Winkler) half-space, as in the plane case we obtain the equation

$$\Delta P_{n+1} = E S_n \Delta\alpha_{n+1}, \quad S_n = S_0/a^{2n}, \quad S_0 = L^2$$

where S_n is the area of the punch at the nth iteration step.

Taking the limit as $n \to \infty$ and integrating, we have

$$P \sim \left(\frac{\alpha}{h}\right)^{\gamma_3}, \quad \gamma_3 = \frac{\ln ba^2}{\ln b} = \frac{3 - D_F}{1 - D_F + D_0} \geqslant 1$$

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